Thm (Billey–Jockush–Stanley)

$$\mathfrak{S}_w = \sum_{p \text{ is pipe-dream of } w} x^{\text{weight}(p)}$$

Nil Coxeter Algebra N_n (over $\mathbb{C}$)

- generators $u_1, \dots, u_{n-1}$
- relations:
  $$\left|\begin{array}{ll} (1) & u_i^2 = 0 \\ (2) & u_i u_j = u_j u_i \text{ if } |i-j| \geq 2 \\ (3) & u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1} \end{array}\right.$$
- linear basis of $N_n = \{u_w : w \in S_n\}$

  Say $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ is reduced decomposition

  define $u_w = u_{i_1} u_{i_2} \cdots u_{i_\ell}$.
- If $v, w \in S_n$ then
  $$u_v \cdot u_w = \begin{cases} u_{vw} & \text{if } \ell(u) + \ell(w) = \ell(uw) \\ 0 & \text{otherwise} \end{cases}$$

$\rightarrow$ Commutative variables $x_{i_1}\ x_{i_2} \cdots x_{i_{n-1}}$ that commute with all u_i-s.

$\rightarrow$ $h_i(x) := 1 + x u_i$

$\rightarrow$ $A_i(x) = h_{n-1}(x) h_{n-2}(x) \cdots h_i(x)$

$\rightarrow$ $\mathcal{S} = A_1(x_1) A_2(x_2) \cdots A_{n-1}(x_{n-1})$

Thm [F.S] $\mathcal{S} = \sum_{w \in S_n} \mathfrak{S}_w(x_1, \ldots, x_n) u_w$

It is not hard to see that this is equivalent to the pipe dream formula.

proof of Thm: Let $\mathcal{S} = \sum_{w \in S_n} \tilde{\mathfrak{S}}_w(x_1, \ldots, x_n) u_w$

we need to check that

(1) $\tilde{\mathfrak{S}}_{w_0}(x_1 \ldots x_n) = x_1^{n-1} \cdots x_1 x_0$

(2) $\tilde{\mathfrak{S}}_w = \partial_i \tilde{\mathfrak{S}}_{w s_i}$ $\quad \ell(s_i w) = \ell(w) + 1$.

It suffices to show (*) $\partial_i S = S u_i$

It suffices to show (**)

$$\partial_i \left(A_i(x_i) A_{i+1}(x_{i+1}) \right) = A_i(x_i) A_{i+1}(x_{i+1}) u_i$$

<u>Lemma 1</u>

(1) $h_i(x) h_i(y) = h_i(x+y)$, $h_i(0) = 1$

(2) $h_i(x) h_j(y) = h_j(y) h_i(x)$ if $|i-j| \geq 2$

(3) $h_i(x) h_{i+1}(x+y) h_i(y) =$

$$= h_{i+1}(x) h_i(x+y) h_{i+1}(x)$$

<u>Yang-Baxter Relations</u>

<u>Lemma 2</u> $A_i(x) A_i(y) = A_i(y) A_i(x)$

<u>Lemma 3</u> $A_i(x) A_{i+1}(y) u_i = A_i(y) A_{i+1}(x) u_i$

Lemma 4 $A_i(x)A_{i+1}(y) - A_i(y)A_{i+1}(x) =$

$= (x-y)A_i(x)A_{i+1}(y)u_i$

Lemma 4 is (**)

Lemma 4 follows from 2 and 3.

It's not hard to check these lemmas.